DELHI PUBLIC SCHOOL,JAMMU

FOUNDATIONAL WORKSHEETS

(2021-2022)

CLASS: XII **SUBJECT: Mathematics**

TOPIC: CONTINUITY AND DIFFERENTIABILITY

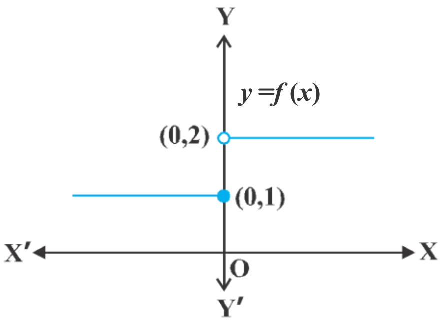
Gist of the lesson:

This chapter is essentially a continuation of our study of differentiation of functions in Class XI. We had learnt to differentiate certain functions like polynomial functions and trigonometric functions. In this chapter, we introduce the very important concepts of continuity, differentiability and relations between them. We will also learn differentiation of inverse trigonometric functions. Further, we introduce a new class of functions called exponential and logarithmic functions. These functions lead to powerful techniques of differentiation. We illustrate certain geometrically obvious conditions through differential calculus. In the process, we will learn some fundamental theorems in this area.

Continuity:

We start the section with two informal examples to get a feel of continuity. Consider the function.

$$(x) = \begin{cases} 1, \text{ if } x \le 0 \\ 2, \text{ if } x > 0 \end{cases}$$



This function is of course defined at every point of the real line.

Graph of this function is given in the Fig

Example 1 Check the continuity of the function *f* given by $f(x) = 2x + 3$ at $x = 1$.

Solution First note that the function is defined at the given point $x = 1$ and its value is 5. Then find the limit of the function at $x = 1$. Clearly

$\lim_{x\to 1} 2x + 3 = f(1) = 5$, Hence f(x) is continuous at x=1.

Example 2 Find all the points of discontinuity of the function *f* defined by

Solution As in the previous example we find that *f* is continuous at all real numbers . The left hand limit of *f* at $x = 1$ is

L.H.L= $\lim_{X\to 1^-} X + 2 = 3$

R.H.L=$\lim_{X\to 1^+} X + 2 = 1$

Since, the left and right hand limits of *f* at $x = 1$ do not coincide, *f* is not continuous at $x = 1$. Hence $x = 1$ is the only point of discontinuity of *f*. The graph of the function is given in the Fig

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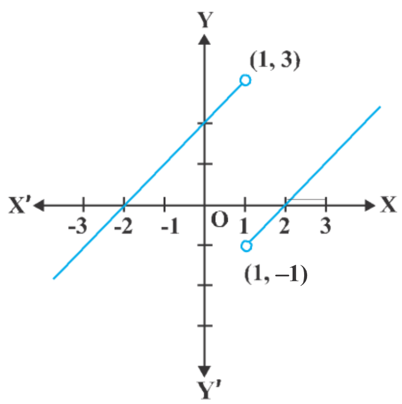
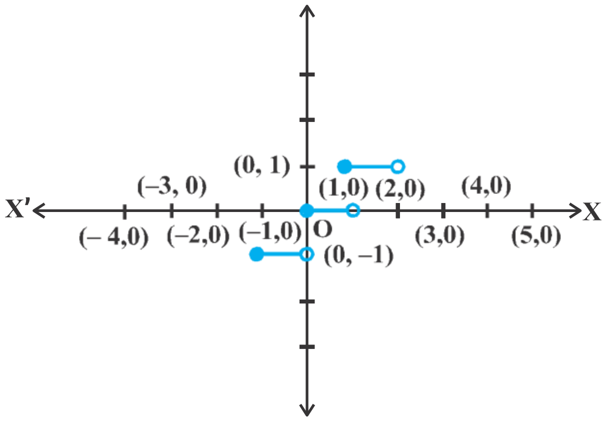


Example 3 Find all the points of discontinuity of the greatest integer function defined by

$$f(x) = [x], \text{ where } [x] \text{ denotes the greatest integer less than or equal to } x.$$

Solution First observe that f is defined for all real numbers. Graph of the function is given in Fig . From the graph it looks like that f is discontinuous at every integral point.



Case 1 Let c be a real number which is not equal to any integer. It is evident from the graph that for all real numbers *close* to c the value of the function is equal to $[c]$; i.e. $\lim_{x\to c} f(x) = \lim_{x\to c}[x] = [c]$, Also $f(c) = [c]$ and hence the function is continuous at all real numbers not equal to integers.

Case 2 Let c be an integer. Then we can find a sufficiently small real number $r > 0$ such that $[c – r] = c – 1$ whereas $[c + r] = c$.

This, in terms of limits means that

$$\lim_{x\to c^-} f(x) = c - 1$$

$$\lim_{x\to c^+} f(x) = c$$

Since these limits cannot be equal to each other for any c, the function is discontinuous at every integral point.

Example 4 Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function.

Solution Observe that the function is defined for every real number. The function f may be thought of as a composition $g \circ h$ of the two functions g and h, where $g(x) = \sin x$ and $h(x) = x^2$. Since both g and h are continuous functions, by Theorem 2, it can be deduced that f is a continuous function.

Example 5 Show that the function f defined by $f(x) = |1 - x + |x||$, where x is any real number, is a continuous function.

Solution Define g by $g(x) = 1 - x + |x|$ and h *by* $h(x) = |x|$ for all real x. Then

$$(h \circ g)(x) = h(g(x))$$

$$= h(1 - x + |x|)$$

$$= |1 - x + |x|| = f(x), \text{ is a continuous function.}$$

Derivative: The rate of change of a quantity y with respect to another quantity x is called the derivative or differential coefficient of y with respect to x.

Differentiation of a Function:

Let f(x) is a function differentiable in an interval [a, b]. That is, at every point of the interval, the derivative of the function exists finitely and is unique. Hence, we may define function g: [a, b] → R, such that, ∀ x ∈ [a, b], g(x) = f'(x).

Fundamental Rules for Differentiation:

(i) $\frac{d}{dx}\{cf(x)\} = c\frac{d}{dx}f(x)$, where c is a constant.

(ii) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$ (sum and difference rule)

(iii) $\frac{d}{dx}\{f(x)\,g(x)\} = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$ (product rule)

Generalization If $u_1, u_2, u_3, \ldots, u_n$ be a function of x, then

$$\frac{d}{dx}(u_1\, u_2\, u_3 \ldots\, u_n) = \left(\frac{du_1}{dx}\right)[u_2 u_3 \ldots u_n] + u_1\left(\frac{du_2}{dx}\right)[u_3 \ldots u_n] + u_1 u_2\left(\frac{du_3}{dx}\right)[u_4 u_5 \ldots u_n] + \ldots + [u_1 u_2 \ldots u_{n-1}]\left(\frac{du_n}{dx}\right)$$

(iv) $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{\{g(x)^2\}}$ (quotient rule)

Different Types of Differentiable Function

Differentiation of Composite Function (Chain Rule)

If f and g are differentiable functions in their domain, then fog is also differentiable and (fog)’ (x) = f’ {g(x)} g’ (x).More easily, if y = f(u) and u = g(x), then dy / dx = dy / du * du / dx.
If y is a function of u, u is a function of v and v is a function of x. Then, dy / dx = dy / du * du / dv * dv / dx.In order to find differential coefficients of complicated expression involving inverse trigonometric functions some substitutions are very helpful, which are listed below.

S. No.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
(ii)	$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $a\cot\theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$
(iv)	$\sqrt{a+x}$ and $\sqrt{a-x}$	$x = a\cos 2\theta$
(v)	$a\sin x + b\cos x$	$a = r\cos\alpha,\ b = r\sin\alpha$
(vi)	$\sqrt{x-\alpha}$ and $\sqrt{\beta - x}$	$x = \alpha\sin^2\theta + \beta\cos^2\theta$
(vii)	$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$

Home Assignment:

CONTINUITY:

1. Discuss the continuity of the following functions:

a) f(x)= $\begin{cases} \frac{1-x^m}{1-x}, if\ x \neq 1 \\ m-1, if\ x = 1 \end{cases}$; mεN, at x=1.

b) f(x)= $\begin{cases} \frac{|x^2-1|}{x-1}, if\ x \neq 1 \\ 4 \quad, if\ x = 1 \end{cases}$; at x=1.

c) f(x)= $\begin{cases} \frac{x-|x|}{x}, if\ x \neq 0 \\ 2 \quad, if\ x = 0 \end{cases}$; at x=0.

d) f(x)= $\begin{cases} \frac{1-\cos 2x}{2x^2}, if\ x \neq 0 \\ 1+3x \quad, if\ x = 0 \end{cases}$; at x=0.

e) f(x)= $|x-1| + |x-2|$ at x=1 and at x=2.

2. If f(x)= $\begin{cases} \frac{5x+|x|}{3x}, if\ x \neq 0 \\ 2, if\ x = 0 \end{cases}$. Show that f(x) is continuous at x=0.

3. For what value of k is the function f given by f(x)= $\begin{cases} 2x+1, x<2 \\ k \quad, x=2 \\ 3x-1, x>2 \end{cases}$ is continuous at x=2.

4. Find the value of k if the function f given by f(x)= $\begin{cases} \frac{1-\cos 2x}{2x^2}, x<0 \\ k, \quad x=0 \\ \frac{x}{|x|}, \quad x>0 \end{cases}$ is continuous at x=0.

5. Find the value of λ so that f(x) = $\begin{cases} \frac{x^2-2x-3}{x+1}, if\ x = -1 \\ \lambda \quad, if\ x = -1 \end{cases}$ is continuous at x= -1.

6. For what value of a is the function f given by f(x)= $\begin{cases} 2x-1, x<2 \\ a \quad, x=2 \\ x+1, x>2 \end{cases}$ is continuous at x=2.

7. Find the value of a for which the function f defined by

f(x)= $\begin{cases} a\sin\frac{\pi}{2}(x+1), x \leq 0 \\ \frac{\tan x - \sin x}{x^3} \quad, x>0 \end{cases}$ is continuous at x=0.

8. A car driver is driving a car on the dangerous path given by

$f(x) = \begin{cases} \frac{1-x^m}{1-x}\ x \neq 0 \\ m-1, x=0 \end{cases}$; ; mεN

Find the dangerous point (point of discontinuity) on the path.

9. Find the values of a and b so that the function f given by

f(x) = $\begin{cases} 1 \quad, if\ x \leq 3 \\ ax+b, if\ 3<x<5 \\ 7, \ if\ x \geq 5 \end{cases}$ is continuous at x=3 and at x=5.

10. The function f(x) is defined as f(x) = $\begin{cases} x^2+ax+b, & 0 \leq x < 2 \\ 3x+2, & 2 \leq x \leq 4 \\ 2ax+5b, & 4 < x \leq 8 \end{cases}$

If f(x) is continuous on [0,8], find the values of a and b.

DIFFERENTIABILITY:

1. Differentiate $\tan^{-1}\left(\frac{1+cosx}{sinx}\right)$ with respect to x.
2. Differentiate w.r.t x : $\cos^{-1}\left(\frac{sinx+cosx}{\sqrt{2}}\right)$.
3. Differentiate w.r.t x:$\cos^{-1}\left[\frac{3x+4\sqrt{1-x^2}}{5}\right]$.
4. Differentiate w.r.t x :$\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$.
5. Find $\frac{dy}{dx}$ at x=1,y=$\frac{\pi}{4}$ if $sin^2y + cos(xy)$= k.
6. If $x^y+y^x=a^b$, then find $\frac{dy}{dx}$.
7. If x= tan($\frac{1}{a}\,logy$),show that $(1+x^2)\frac{d^2y}{dx^2}$+(2x-a)$\frac{dy}{dx}$=0.
8. If $\sqrt{1-x^2}$ +$\sqrt{1-y^2}$ = a(x-y) , prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.
9. Differentiate $\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+36^x}\right)$ with respect to x.